

A TOPOLOGICAL INVARIANT OF LINE ARRANGEMENTS

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ABSTRACT. We define a new topological invariant of line arrangements in the complex projective plane. This invariant is a root of unity defined under some combinatorial restrictions for arrangements endowed with some special torsion character on the fundamental group of their complements. It is derived from the peripheral structure on the group induced by the inclusion map of the boundary of a tubular neighborhood in the exterior of the arrangement. By similarity with knot theory, it can be viewed as an analogue of linking numbers. This is an orientation-preserving invariant for ordered arrangements. We give an explicit method to compute the invariant from the equations of the arrangement, by using wiring diagrams introduced by Arvola, that encode the braid monodromy. Moreover, this invariant is a crucial ingredient to compute the depth of a character satisfying some resonant conditions, and complete the existent methods by Libgober and the first author. Finally, we compute the invariant for extended MacLane arrangements with an additional line and observe that it takes different values for the deformation classes.

INTRODUCTION

The influence of the combinatorial data on the topology of a projective line arrangement is not at present well understood. In the eighties, Orlik and Solomon [15] showed that the cohomology ring of the complement of an arrangement is determined by the description of the incidence relations between the multiple points. This is not true for the deformation classes, as it was shown for MacLane combinatorics [14, 6]. Rybnikov [16, 17, 4] constructed a pair of complex line arrangements with the same combinatorics but whose fundamental groups are not isomorphic. This illustrates that the combinatorics of an arrangement \mathcal{A} does not determine in general the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{A})$. Other examples were exhibited by the first author, Carmona, Cogolludo and Marco by using the braid monodromy [3].

One of the strongest invariant of the topology of an arrangement is the fundamental group of its complement, which can be computed using Zariski-van Kampen method, see also the specific approach by Arvola [5]. Even when the fundamental group can be computed, it is very difficult to handle directly with. This is why a number of invariants derived from the group have emerged, as the Alexander

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invariant and the characteristic varieties. These invariants can be computed from the fundamental group, but the task can be endless for present computers for most arrangements. A weaker invariant is the one-variable Alexander polynomial (also known as characteristic polynomial of the monodromy of the Milnor fiber) which can be computed directly from the fundamental group or by a general method available for any projective curve. Though there are some partial results, it is still unknown whether characteristic varieties or Alexander polynomial are combinatorially determined.

Using a method by Libgober [13] it is possible to compute *most* irreducible components of characteristic varieties (only some isolated points may fail to be found). A method to compute these extra components can be found in [2].

The *boundary manifold* $B_{\mathcal{A}}$, defined as the common boundary of a regular neighbourhood of \mathcal{A} and its exterior $E_{\mathcal{A}}$, is a graph 3-manifold whose structure is determined by the combinatorics of \mathcal{A} . In the present paper, we construct a new topological invariant of line arrangements derived from the *peripheral structure* $\pi_1(B_{\mathcal{A}}) \rightarrow \pi_1(E_{\mathcal{A}})$, induced by the inclusion map, and previously studied in [11]. This invariant is a root of unity defined for triples composed by an arrangement, a torsion character in \mathbb{C}^* and a cycle in the incidence graph of the arrangement; there are combinatorial restrictions for the availability of this invariant, in particular the cycle must be non trivial and satisfy some resonant conditions. As it was remarked in [12], it is in fact extracted from the homological reduction $i_* : H_1(B_{\mathcal{A}}) \rightarrow H_1(E_{\mathcal{A}})$, which is a more tractable object, and corresponds to the value of the character on certain homology classes of the boundary manifold, viewed in $E_{\mathcal{A}}$ through the inclusion map. This construction has similarities with knot theory, and the invariant is a sort of analogue of linking numbers. As we state in the first main result, Theorem 2.1, it is an *ordered-oriented* topological invariant.

As a second main contribution of this work, we give an explicit method to compute the invariant, in terms of braid monodromy. We use *braided wiring diagrams* introduced by Arvola [5] (see also Suciu-Cohen [7]) to encode the braid monodromy relative to a generic projection of the arrangement. Note that this invariant is most probably of algebraic nature, even though our computations are topological.

It appears that this invariant is crucial for the computation of the *quasi-projective* depth of a (torsion) character in [2]. The knowledge of depths of all characters is equivalent to the knowledge of characteristic varieties; the depth can be decomposed into a *projective term* and a *quasi-projective term*, vanishing for characters that ramify along all the lines. An algorithm to compute the projective part was given by Libgober (see [13] and also [2] for details). An explicit way to compute the quasi-projective depth of resonant (torsion) characters is given in [2], and it happens that the invariant in this paper is crucial for that method. Hence, our invariant may help to find examples of combinatorially equivalent arrangements with different structure for their characteristic varieties, though we have failed till now in finding such arrangements. But as we show in this paper, this invariant is interesting in its own.

We compute the invariant for MacLane arrangements with an additional line, and observe that it takes different values for the two deformation classes. This shows that it provides information

on their topologies, not contained in the combinatorics. In particular, there is no ordered-oriented homeomorphism between both realizations; note that this fact is a consequence of the same result for MacLane arrangements (as shown by Rybnikov).

In a forthcoming paper, the third author present new examples of *Zariski pairs* of arrangements with 12 lines (of different nature of previous examples), where this invariant is the unique known one to show that they have not the same topological type.

In §1 we introduce the combinatorial and topological objects to be used in the paper, namely the concept of inner-cyclic triplets. In §2, we define the invariant of the realization of an inner-cyclic triplet and we prove its topological invariance. In §3, the relationship of the invariant with characteristic varieties is described. The topological computation of the invariant via wiring diagrams and the results of [11] is in §4. Finally, §5 computes the invariant for the two ordered realizations of the extended MacLane combinatorics.

1. INNER CYCLIC TRIPLETS

1.1. Combinatorics.

In what follows, a brief reminder on line combinatorics is given, see [3] for details. We also introduce the notion of *inner cyclic*.

Definition 1.1. A *combinatorial type*, or simply a *(line) combinatorics*, is a couple $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, where \mathcal{L} is a finite set and $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$, satisfying that:

- For all $p \in \mathcal{P}$, $\#p \geq 2$;
- For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, $\exists! p \in \mathcal{P}$ such that $\ell_1, \ell_2 \in p$.

An *ordered combinatorics* \mathcal{C} is a combinatorics where \mathcal{L} is an ordered set.

This notion encodes the intersection pattern of a collection of lines in a projective planes, see §1.2, where the relation \in corresponds to the dual plane. There are several ways to encode a line combinatorics.

Definition 1.2. The *incidence graph* $\Gamma_{\mathcal{C}}$ of a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ is a non-oriented bipartite graph where the set of vertices $V(\mathcal{C})$ decomposes as $V_P(\mathcal{C}) \amalg V_L(\mathcal{C})$, with:

$$V_P(\mathcal{C}) = \{v_p \mid p \in \mathcal{P}\}, \quad \text{and} \quad V_L(\mathcal{C}) = \{v_\ell \mid \ell \in \mathcal{L}\}.$$

The vertices of $V_P(\mathcal{C})$ are called *point-vertices* and those of $V_L(\mathcal{C})$ are called *line-vertices*. An edge of $\Gamma_{\mathcal{C}}$ joins v_ℓ to v_p if and only if $\ell \in p$. Such an edge is denoted by $e(\ell, p)$.

For a line arrangement in the projective plane the incidence graph is the dual graph of the divisor obtained by the preimage of the line arrangement in the blowing-up of the projective plane along the set of multiples points of the arrangement, see §1.2.

Definition 1.3. A *character* on a line combinatorics $(\mathcal{L}, \mathcal{P})$ is a map $\xi : \mathcal{L} \rightarrow \mathbb{C}^*$ such that

$$(1.1) \quad \prod_{\ell \in \mathcal{L}} \xi(\ell) = 1.$$

A *torsion character* on a line combinatorics $(\mathcal{L}, \mathcal{P})$ is a character ξ where for all $\ell \in \mathcal{L}$, $\xi(\ell)$ is a root of unity.

Namely, we are associating a non-zero complex number to each element of \mathcal{L} , such that the product of all of them equals 1. These characters have a cohomological meaning for line arrangements in the complex projective plane.

Definition 1.4. Let ξ be a character on a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$. For each $p \in \mathcal{P}$, we define $\xi(p) := \prod_{\ell \in p} \xi(\ell)$.

A *cycle* of $\Gamma_{\mathcal{C}}$ is an element of $H_1(\Gamma_{\mathcal{C}})$. Finally, we introduce the main object of this work.

Definition 1.5. An *inner cyclic triplet* $(\mathcal{C}, \xi, \gamma)$ is a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, a torsion character ξ on \mathcal{L} and a cycle γ of $\Gamma_{\mathcal{C}}$ such that:

- (1) for all line-vertex v_{ℓ} of γ , $\xi(\ell) = 1$,
- (2) for all point-vertex v_p of γ , and for all $\ell \in p$, $\xi(\ell) = 1$,
- (3) for all $p \in \mathcal{P}$, with $v_{\ell} \in \gamma$, $\xi(p) = 1$.

The above conditions can be understood in a shorter way: all the vertices of γ and all their neighbors come from elements $m \in V(\mathcal{C})$ such that $\chi(m) = 1$.

1.2. Realization.

Let \mathcal{A} be a line arrangement in $\mathbb{CP}^2 \equiv \mathbb{P}^2$ and let $\mathcal{P}_{\mathcal{A}}$ be the set of multiple points of \mathcal{A} ; then $\mathcal{C}_{\mathcal{A}} := (\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ is the *combinatorics* of \mathcal{A} . Given a combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, a *complex realization* of \mathcal{C} is a line arrangement \mathcal{A} in \mathbb{P}^2 such that its combinatorics agrees with \mathcal{C} . An *ordered realization* of an ordered combinatorics is defined accordingly. The existence of realizations of a combinatorics depends on the field \mathbb{K} .

Given a line arrangement \mathcal{A} in \mathbb{P}^2 , or more generally a set of irreducible curves \mathcal{A} in a projective surface X , we will denote $\bigcup \mathcal{A}$ the union of those curves.

Remark 1.6. Let \mathcal{A} be a line arrangement in \mathbb{P}^2 with combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ (note that $\mathcal{L} = \mathcal{A}$). For $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$ the notion $\ell \in p$ can be understood in the dual plane $\check{\mathbb{P}}^2$. For convenience we also use the notation $p \in \ell$. Let $\pi : \widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ be the composition of the blow-ups of the points in \mathcal{P} ; then $\tilde{\mathcal{A}} := \pi^{-1}(\mathcal{A})$ defines a normal crossing divisor in $\widehat{\mathbb{P}}^2$ whose dual graph is exactly $\Gamma_{\mathcal{C}}$. Note also that $M_{\mathcal{A}} := \mathbb{P}^2 \setminus \bigcup \mathcal{A} = \widehat{\mathbb{P}}^2 \setminus \bigcup \tilde{\mathcal{A}}$. Let us identify each $v_{\ell} \in V_L(\mathcal{C})$ with a meridian of ℓ as an element in $H_1(M_{\mathcal{A}}; \mathbb{Z})$. Note that $V_L(\mathcal{C})$ generates $H_1(M_{\mathcal{A}}; \mathbb{Z})$ and the only relation satisfied by them is (1.1).

Moreover, if $v_p \in V_P(\mathcal{C})$ is identified with a meridian of $\pi^{-1}(P)$ as element of $H_1(M_{\mathcal{A}}; \mathbb{Z})$, then, the equality

$$(1.2) \quad v_p = \prod_{p \in \ell} v_\ell = \prod_{\ell \in p} v_\ell$$

holds. Via this identification the space of characters of \mathcal{C} coincides with

$$H^1(M_{\mathcal{A}}; \mathbb{C}^*) = \text{Hom}(H_1(M_{\mathcal{A}}; \mathbb{Z}), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\sharp \mathcal{A}-1}.$$

Equation (1.2) agrees with Definition 1.4.

The space $M_{\mathcal{A}}$ is not compact and this may cause some trouble. To avoid this complication, let $\text{Tub}(\mathcal{A})$ be a compact regular neighbourhood of \mathcal{A} in \mathbb{P}^2 , and let $E_{\mathcal{A}} := \overline{\mathbb{P}^2} \setminus \overline{\text{Tub}(\mathcal{A})}$ be the *exterior* of the arrangement. This is an oriented 4-manifold with boundary such that the inclusion $E_{\mathcal{A}} \hookrightarrow M_{\mathcal{A}}$ is a homotopy equivalence and in particular $H_1(E_{\mathcal{A}}; \mathbb{Z}) \cong H_1(M_{\mathcal{A}}; \mathbb{Z})$. As we have seen, if $\sharp \mathcal{A} = n + 1$, then $H_1(E_{\mathcal{A}}; \mathbb{Z}) \cong \mathbb{Z}^n$ and it is freely generated by the meridians of any subset of n lines in \mathcal{A} .

Notation 1.7. We use the notation $\Gamma_{\mathcal{A}}$ for the incidence graph of $\Gamma_{\mathcal{C}}$.

Definition 1.8. Let $(\mathcal{C}, \xi, \gamma)$ be a triplet, where \mathcal{C} is an ordered combinatorics, ξ a character and γ a circular cycle of $\Gamma_{\mathcal{C}}$. A *realization* of $(\mathcal{C}, \xi, \gamma)$ is a triplet $(\mathcal{A}, \xi_{\mathcal{A}}, \gamma_{\mathcal{A}})$:

- An ordered realization \mathcal{A} of \mathcal{C} ;
- A character $\xi_{\mathcal{A}} : H_1(E_{\mathcal{A}}; \mathbb{Z}) \rightarrow \mathbb{C}^*$ such that $\xi_{\mathcal{A}}(v_\ell) = \xi(\ell)$ under the identification of Remark 1.6.
- A cycle $\gamma_{\mathcal{A}}$ in $\Gamma_{\mathcal{A}}$ which coincides with γ via the natural identification $\Gamma_{\mathcal{A}} \equiv \Gamma_{\mathcal{C}}$.

Due to this natural identifications we usually drop the subindex \mathcal{A} . If $(\mathcal{C}, \xi, \gamma)$ is an inner cyclic triplet, then the realization $(\mathcal{A}, \xi, \gamma)$ is *inner cyclic*.

1.3. Regular neighborhoods.

Let \mathcal{A} be a line arrangement with combinatorics \mathcal{C} . Let us describe how to construct a compact regular neighbourhood $\text{Tub}(\mathcal{A})$. There are several ways to define it, see [9, 10], and they produce isotopic results.

For each $\ell \in \mathcal{A}$ we consider a tubular neighborhood $\text{Tub}(\ell)$ of $\ell \subset \mathbb{P}^2$ and for each $p \in \mathcal{P}$ we consider a closed 4-ball \mathbb{B}_p centered at p .

Definition 1.9. We say that the set $\{\text{Tub}(\ell) \mid \ell \in \mathcal{A}\} \cup \{\mathbb{B}_p \mid p \in \mathcal{P}\}$ is a *compatible system of neighborhoods in \mathcal{A}* if $\forall \ell_1 \neq \ell_2$ we have

$$\text{Tub}(\ell_1) \cap \text{Tub}(\ell_2) = \mathring{\mathbb{B}}_p, \quad p = \ell_1 \cap \ell_2$$

and the balls are pairwise disjoint. The union of these neighborhoods is a *regular neighborhood* of \mathcal{A} . Given such a system, for each ℓ we define the *holed neighborhood*

$$\mathcal{N}(\ell) := \overline{\text{Tub}(\ell) \setminus \bigcup_{p \in \ell} \mathbb{B}_p}.$$

For each $\ell \in \mathcal{A}$ we denote by

$$\check{\ell} := \overline{\ell \setminus \bigcup_{p \in \ell} \mathbb{B}_p}.$$

This is a punctured sphere (which as many punctures as the number of multiple points in ℓ) and the space $\mathcal{N}(\ell)$ is (non-naturally) homeomorphic to $\check{\ell} \times D^2$, where D^2 is a closed disk in \mathbb{C} .

Let us consider now $\pi : \widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$. For each $p \in \mathcal{P}$, $\pi^{-1}(\mathbb{B}_p)$ is a tubular neighborhood of the rational curve $E_p := \pi^{-1}(p)$; this is a locally trivial D^2 -bundle, but not trivial. Note that $\pi^{-1}(\mathcal{N}(\ell))$ is naturally isomorphic to $\mathcal{N}(\ell)$ and that $\text{Tub}(\tilde{\mathcal{A}}) := \pi^{-1}(\text{Tub}(\mathcal{A}))$ is a regular neighborhood of $\tilde{\mathcal{A}}$ obtained by plumbing the tubular neighborhood of its irreducible components.

If $p \in \ell$ then $V_{\ell,p} := \mathcal{N}(\ell) \cap \mathbb{B}_p$ is a tubular neighborhood of the trivial knot $\ell \cap \partial \mathbb{B}_p \subset \partial \mathbb{B}_p$, i.e. a solid torus.

The manifolds $E_{\mathcal{A}}$ and $\text{Tub}(\mathcal{A})$ share their boundaries $B_{\mathcal{A}} = E_{\mathcal{A}} \cap \text{Tub}(\mathcal{A})$ and note also that $B_{\mathcal{A}}$ can be identified with the boundary of $\text{Tub}(\tilde{\mathcal{A}})$. This manifold is a graph manifold obtained by gluing the following pieces:

$$B_{\ell} := \overline{\partial \mathcal{N}(\ell) \setminus \bigcup_{p \in \ell} V_{\ell,p}}, \quad B_p := \overline{\partial \mathbb{B}_p \setminus \bigcup_{\ell \in p} V_{\ell,p}}.$$

Note also that $B_{\mathcal{A}} \hookrightarrow \text{Tub}(\mathcal{A}) \setminus \bigcup \mathcal{A}$ is a homotopy equivalence.

1.4. Nearby cycles.

Let \mathcal{A} be an arrangement, and γ be a circular cycle of $\Gamma_{\mathcal{A}}$. The *support* of γ is defined as:

$$\text{supp}(\gamma) = \{\ell \in \mathcal{A} \mid v_{\ell} \in \gamma\} = \{\ell_1, \dots, \ell_r\}.$$

with cyclic order $\ell_1 < \dots < \ell_r$ and $\ell_{r+1} := \ell_1$. Let $p_j := \ell_j \cap \ell_{j+1}$, $j = 1, \dots, r$.

Definition 1.10. An *embedding* of γ in \mathcal{A} is a simple closed loop $r(\gamma) \subset \bigcup \mathcal{A}$ defined as follows. Take a point $q_j \in \check{\ell}_j$ ($q_{r+1} := q_1$), $j = 1, \dots, r$. We denote by p_j^{α} a point in $\ell_j \cap \mathbb{B}_{p_j}$ and by p_j^{ω} a point in $\ell_{j+1} \cap \mathbb{B}_{p_j}$. Let ρ_j^{α} be a radius in ℓ_j from p_j to p_j^{α} and ρ_j^{ω} be a radius in ℓ_{j+1} from p_j to p_j^{ω} . Pick up arbitrary simple paths α_j from q_j to p_j^{α} in $\check{\ell}_j$, $j = 1, \dots, r$, and ω_j from p_{j-1}^{ω} to q_j , $j = 2, \dots, r+1$. Then:

$$r(\gamma) := \alpha_1 \cdot (\rho_1^{\alpha})^{-1} \cdot \rho_1^{\omega} \cdot \omega_2 \cdot \alpha_2 \cdot \dots \cdot (\rho_r^{\alpha})^{-1} \cdot \rho_r^{\omega} \cdot \omega_{r+1}.$$

Definition 1.11. A *nearby cycle* $\tilde{\gamma}$ associated with γ is a smooth path in $B_{\mathcal{A}}$, homologous to an embedding $r(\gamma)$ of γ in $\text{Tub}(\mathcal{A})$, lying in

$$\left(\bigcup_{j=1}^r \mathcal{N}(\ell_j) \cup \bigcup_{j=1}^r \mathbb{B}_{p_j} \right) \setminus \bigcup \mathcal{A}.$$

Remark 1.12. Note that if $p \neq p_j$ a nearby cycle $\tilde{\gamma}$ can intersect \mathbb{B}_p only at some $V_{\ell_k, p}$ for some k ; there are always nearby cycles which do not intersect \mathbb{B}_p for $p \neq p_j$.

2. INVARIANT

Let $(\mathcal{C}, \xi, \gamma)$ be a inner-cyclic triplet; and suppose that $(\mathcal{A}, \xi, \gamma)$ is a realization. Denote by i the inclusion map of the boundary manifold in the exterior, i.e., $i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$. We consider the following composition map:

$$\chi_{(\mathcal{A}, \xi)} : H_1(B_{\mathcal{A}}) \xrightarrow{i_*} H_1(E_{\mathcal{A}}) \xrightarrow{\xi} \mathbb{C}^*$$

Let $\tilde{\gamma}$ be a nearby cycle associated with γ . As we are going to prove in Lemma 2.2, we can define

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) := \chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}).$$

Our main goal is to prove that this is a topological invariant, as we state here.

Theorem 2.1 (Main result). *If $(\mathcal{A}, \xi, \gamma)$ and $(\mathcal{A}', \xi, \gamma)$ are two inner-cyclic realizations of $(\mathcal{C}, \xi, \gamma)$ with the same (oriented and ordered) topological type, then*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \mathcal{I}(\mathcal{A}', \xi, \gamma).$$

The aim of this section is to prove Theorem 2.1. The first step is to prove the following Lemma which shows that for a given inner-cyclic realization $(\mathcal{A}, \xi, \gamma)$, the image of a nearby cycle $\tilde{\gamma}$ by $\chi_{(\mathcal{A}, \xi)}$ depends only in γ .

Lemma 2.2. *Let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic realization. If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are nearby cycles associated with γ , then:*

$$\chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}) = \chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}').$$

Proof. The choice of a nearby cycle in $B_{\mathcal{A}}$ associated with γ depends first on the embedding $r(\gamma)$ in \mathcal{A} , see Definition 1.10 and use the notation inside. There is some freedom in the choice of $r(\gamma)$; in $r(\gamma) \cap \ell_j$ we can add some meridians around the multiple points of \mathcal{A} in ℓ_j ; recall that for any such multiple point p we have $\xi(p) = 1$ by Definition 1.5(3).

Once $r(\gamma)$ has been chosen, $\tilde{\gamma}$ lies in $B_{\mathcal{A}}$, and is homologous to $r(\gamma)$ in $\text{Tub}(\mathcal{A})$. Two different representants will differ by the meridians v_{ℓ_j} and by cycles in $\mathbb{B}_{p_j} \setminus \mathcal{A}$; these ones are linear combinations of meridians v_{ℓ} for $\ell \in p_j$.

Summarizing the difference of two nearby cycles is a linear combination of meridians v_ℓ , where either $\ell \in \text{supp}(\gamma)$ or $\ell \in p_j$, and meridians v_p , $p \in \ell_j$. By Definition 1.5 all these meridians are in $\ker \xi$ and the result follows. \square

We have proved with this Lemma that $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is well-defined for a particular \mathcal{A} . The following result studies the behavior of this invariant under orientation and order preserving homeomorphisms. We prove that there are nearby cycles of the first realization which are sent to nearby cycles of the second realization; note that we do not need the stronger result that would say that the image of *any* nearby cycle is a nearby cycle.

Lemma 2.3. *Let $(\mathcal{C}, \xi, \gamma)$ be an ordered inner-cyclic triplet. Suppose that $(\mathcal{A}, \xi, \gamma)$ and $(\mathcal{A}', \xi, \gamma)$ are two ordered realizations of $(\mathcal{C}, \xi, \gamma)$ such that there exists a homeomorphism $\phi : (\mathbb{P}^2, \mathcal{A}) \rightarrow (\mathbb{P}^2, \mathcal{A}')$ preserving orders and orientations of the lines. Then there is a nearby cycle $\tilde{\gamma}$ associated to γ in $B_{\mathcal{A}}$, such that $\phi(\tilde{\gamma})$ is isotopic to a nearby cycle associated with γ in $B_{\mathcal{A}'}$.*

Proof. If ℓ is a line in \mathcal{A} , ℓ' is the corresponding line in \mathcal{A}' ; we use the same convention for multiple points.

We start fixing a regular neighborhood $\text{Tub}(\mathcal{A}')$ obtained as a union of $\mathbb{B}_{p'}$ and $\mathcal{N}(\ell')$. Since ϕ is a homeomorphism we can construct a regular neighborhood $\text{Tub}(\mathcal{A})$ obtained as a union of \mathbb{B}_p and $\mathcal{N}(\ell)$, such that $\phi(\mathbb{B}_p) \subset \mathbb{B}_{p'}$ and $\phi(\mathcal{N}(\ell)) \subset \mathcal{N}(\ell')$.

Let us choose an embedding $r(\gamma)$ in \mathcal{A} such that $r(\gamma)$ is disjoint from \mathbb{B}_p for $p \neq p_1, \dots, p_r$ which is always possible. Moreover we can construct a continuous family of nearby cycles $\tilde{\gamma}_t$, $t \in (0, 1]$ (converging to $r(\gamma)$ when $t \rightarrow 0$) and such that this family is still disjoint with \mathbb{B}_p for $p \neq p_1, \dots, p_r$. Then, $\phi(\tilde{\gamma}_1)$ is a nearby cycle associated with γ which can be deformed to a nearby cycle in $B_{\mathcal{A}'}$. \square

Remark 2.4. The nearby cycles used in Section 4 are different of the one used in this proof. Those cycles may intersect the boundary of $\partial\mathcal{N}$, and this is important for the computation of the invariant, but it does not affect to our goal.

Proof of Theorem 2.1. Suppose that there exists $\phi : (\mathbb{P}^2, \mathcal{A}) \rightarrow (\mathbb{P}^2, \mathcal{A}')$ preserving orders and orientations of the lines. Hence ϕ induces an isomorphism $\phi_* : H_1(\mathbb{P}^2 \setminus \mathcal{A}) \rightarrow H_1(\mathbb{P}^2 \setminus \mathcal{A}')$ preserving ξ . Let $\tilde{\gamma}$ be an arbitrary nearby cycle associated with γ . By Lemma 2.2, $\mathcal{I}(\mathcal{A}, \xi, \gamma) = \chi_{(\mathcal{A}, \xi)}(\tilde{\gamma})$ depends only on γ . We can then assume that $\tilde{\gamma}$ is one of the particular nearby cycle considered in Lemma 2.3. But this lemma implies that $\phi(\tilde{\gamma})$ is isotopic to a nearby cycle $\tilde{\gamma}'$ associated to γ in $B_{\mathcal{A}'}$. It follows that $\chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}') = \chi_{(\mathcal{A}, \xi)}(\tilde{\gamma})$, and we conclude using Lemma 2.2. \square

Proposition 2.5. *Let $(\mathcal{A}, \xi, \gamma)$ and $(\overline{\mathcal{A}}, \xi, \gamma)$ be two inner cyclic complex conjugated realization arrangements. We have then:*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma)^{-1} = \mathcal{I}(\overline{\mathcal{A}}, \xi, \gamma).$$

Proof. The complex conjugation $(\mathbb{P}^2, \mathcal{A}) \rightarrow (\mathbb{P}^2, \overline{\mathcal{A}})$ is an ordered homeomorphism which preserves the orientation of \mathbb{P}^2 but exchanges the orientation of the line. For this reason the action of the homeomorphism sends ξ to ξ^{-1} (since the conjugation sends a meridian v_ℓ to (v_ℓ^{-1}) , with complex notation). The result follows from the fact that a nearby cycle for $(\overline{\mathcal{A}}, \xi, \gamma)$ it is also a nearby cycle for $(\overline{\mathcal{A}}, \xi^{-1}, \gamma)$. \square

Corollary 2.6. *If \mathcal{A} is a complexified real arrangement, that is the complexification of a line arrangement defined on \mathbb{RP}^2 , then:*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) \in \{-1, 1\}.$$

The same result occurs if there is a continuous path of combinatorially equivalent line arrangements connecting \mathcal{A} and $\overline{\mathcal{A}}$.

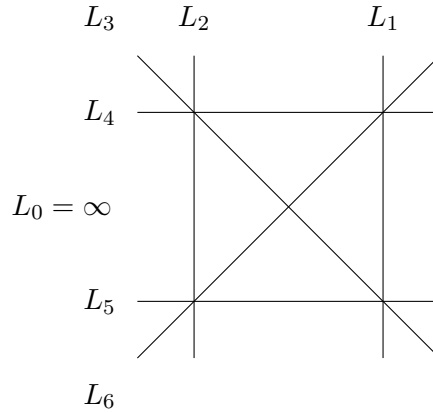


FIGURE 1. The Ceva-7 arrangement \mathcal{C}_7

Example 2.7. Let $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ be the combinatorics defined by $\mathcal{L} = \{L_0, \dots, L_6\}$ and

$$\begin{aligned} \mathcal{P} = \{ \{L_0, L_1, L_2\}, \{L_0, L_3\}, \{L_0, L_4, L_5\}, \{L_0, L_6\}, \{L_1, L_3, L_5\}, \\ \{L_1, L_4, L_6\}, \{L_2, L_3, L_4\}, \{L_2, L_5, L_6\}, \{L_3, L_6\} \}. \end{aligned}$$

The following triplet $(\mathcal{C}, \xi, \gamma)$ is inner-cyclic:

- ξ is the character on $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ defined by:

$$(L_0, L_1, \dots, L_6) \mapsto (1, -1, -1, 1, -1, -1, 1).$$

- γ is the cycle of $\Gamma_{\mathcal{C}}$ defined by $v_{L_0} \rightarrow v_{P_{0,3}} \rightarrow v_{L_3} \rightarrow v_{P_{3,6}} \rightarrow v_{L_6} \rightarrow v_{P_{0,6}} \rightarrow v_{L_0}$.

The arrangement \mathcal{C}_7 pictured in Figure 1 is –up to projective transformation– the only one realization of the combinatorics \mathcal{C} . There is a nearby cycle $\tilde{\gamma}$ associated with the cycle γ such that its image by the map $i_* : H_1(B_{\mathcal{C}_7}) \rightarrow H_1(E_{\mathcal{C}_7})$ is $-v_5$ (an algorithm to compute is given in Section 4). Then we have:

$$\mathcal{I}(\mathcal{C}_7, \xi, \gamma), \xi) = \xi(-v_5) = -1.$$

3. CHARACTERISTIC VARIETIES

The characteristic varieties of an arrangement can be defined as the jumping loci of twisted cohomology of rank one local system. They only depend on its fundamental group. For $\xi \in H^1(M_{\mathcal{A}}; \mathbb{C}^*)$, let \mathcal{L}_{ξ} be local system of coefficients defined by ξ .

Definition 3.1. The *characteristic varieties* of an arrangement \mathcal{A} are:

$$\mathcal{V}_k(\mathcal{A}) = \{ \xi \in \mathbb{T}(\mathcal{A}) \mid \dim_{\mathbb{C}} (H^1(M_{\mathcal{A}}; \mathcal{L}_{\xi})) \geq k \}.$$

Definition 3.2. The *depth* of a character $\xi \in \mathbb{T}(\mathcal{A})$ is:

$$\text{depth}(\xi) = \max \{ k \in \mathbb{N} \mid \xi \in \mathcal{V}_k \} = \dim_{\mathbb{C}} H^1(M_{\mathcal{A}}; \mathcal{L}_{\xi}).$$

Hence for the study of characteristic varieties, we need to be able to compute the twisted cohomology spaces of $M_{\mathcal{A}}$.

3.1. Geometric interpretation of the notion of inner cyclic.

Let us recall that $\pi : \hat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ is the blow-up of \mathbb{P}^2 over the points of \mathcal{P} ; the main goal of the construction of $\hat{\mathbb{P}}^2$ is to obtain $\tilde{\mathcal{A}} := \pi^{-1}(\mathcal{A})$ as a normal crossing divisor (in fact, we need only to blow up the points of multiplicity at least three, but it is harmless to do extra blow-ups). By construction, we have that $\hat{\mathbb{P}}^2 \setminus \bigcup \hat{\mathcal{A}} \equiv E_{\mathcal{A}}$, then a character ξ on $\pi_1(E_{\mathcal{A}})$ can be view as a character on $\pi_1(\hat{\mathbb{P}}^2 \setminus \bigcup \hat{\mathcal{A}})$ (also noted ξ). Let $\hat{\Gamma}_{\mathcal{A}}$ be the dual graph of $\hat{\mathcal{A}}$.

Definition 3.3. A component $H \in \hat{\mathcal{A}}$ is *unramified* for the character ξ if $\xi(v_H) = 1$. It is *inner unramified* for ξ if it is unramified and all its neighbours (in $\hat{\Gamma}_{\mathcal{A}}$) too. The set of all the inner unramified components of $\hat{\mathcal{A}}$ is denoted by $\mathcal{U}_{\xi} \subset \hat{\mathcal{A}}$. The dual graph of $\hat{\mathcal{A}}$ of inner unramified components is denoted by $\hat{\Gamma}_{\mathcal{U}_{\xi}}$.

Definition 3.4. Let \mathcal{A} be an arrangement. A character ξ is *inner-cyclic* if it is torsion and $b_1(\hat{\Gamma}_{\mathcal{U}_{\xi}}) > 0$.

Proposition 3.5. An inner-cyclic arrangement is the data of a triple $(\mathcal{A}, \xi, \gamma)$, where ξ is an inner-cyclic character on $E_{\mathcal{A}}$ and γ a cycle of $\hat{\Gamma}_{\mathcal{U}_{\xi}}$.

Remark 3.6. With this point of view, a nearby cycle is a cycle leaving in the boundary of a regular neighbourhood of the union of inner unramified components.

3.2. Quasi-projective depth.

The depth of a torsion character is decomposed in two terms, the projective and quasi-projective depth, see [2, 12]. Summarizing, if ξ is a torsion character of order n , then there is an n -fold unbranched ramified quasi-projective cover $\rho : E_{\mathcal{A}}^{\xi} \rightarrow E_{\mathcal{A}}$ where $\sigma : X^{\xi} \rightarrow X^{\xi}$ generates the deck group of the cover. There is a natural isomorphism from $H^1(E_{\mathcal{A}}; \mathcal{L}_{\xi})$ to the eigenspace of $H^1(X^{\xi}; \mathbb{C})$ for σ with eigenvalue $e^{\frac{2i\pi}{n}}$.

Let $\bar{\rho} : X^{\xi} \rightarrow \widehat{\mathbb{P}}^2$ be a smooth model of the projectivization of ρ , where $\bar{\sigma}$ generates the deck group. The inclusion $j_N : E_{\mathcal{A}}^{\xi} \hookrightarrow X^{\xi}$ induces an injection

$$j_N^* : H^1(X^{\xi}; \mathbb{C}) \rightarrow H^1(E_{\mathcal{A}}^{\xi}; \mathbb{C}).$$

We denote by $j_{N,\xi}^*$ the restriction of j_N^* to the eigenspaces for $\sigma, \bar{\sigma}$ with eigenvalue $e^{\frac{2i\pi}{n}}$.

Definition 3.7. Let ξ be a torsion character on $\pi_1(E_{\mathcal{A}})$. The *projective depth* of ξ is $\dim \operatorname{Im} j_{N,\xi}^*$ while the *quasi-projective depth* of ξ is:

$$\overline{\operatorname{depth}}(\xi) = \dim \operatorname{coker}(j_{N,\xi}^*).$$

Note that there are known formulas for the computation of the projective depth given by Libgober [13], see [1] for details. Moreover, there is a *finite-time* algorithm to compute this projective depth for any character. We explain now the method of [1] to compute the quasi-projective depth.

We are going to construct a matrix for a *twisted* hermitian intersection \cdot formulas in a vector space having as base the elements of \mathcal{U}_{ξ} ; for an arbitrary order of this basis we will consider a square matrix A_{ξ} of size $\#\mathcal{U}_{\xi}$, where coefficients are indexed by elements of \mathcal{U}_{ξ} . This matrix will depend on the choice of a maximal tree $\mathcal{T}_{\mathcal{U}_{\xi}}$ of $\widehat{\Gamma}_{\mathcal{U}_{\xi}}$ (maybe a maximal forest, since $\widehat{\Gamma}_{\mathcal{U}_{\xi}}$ is non necessarily connected). For each (oriented) edge e not in $\mathcal{T}_{\mathcal{U}_{\xi}}$ we consider a cycle γ_e consisting on e and a linear chain of $\widehat{\Gamma}_{\mathcal{U}_{\xi}}$ connecting the final end of e with its starting point. Let us denote:

$$\chi(e) := \begin{cases} \mathcal{I}(\mathcal{A}, \xi, \gamma) & \text{if } e \notin \mathcal{T}_{\mathcal{U}_{\xi}} \\ 1 & \text{otherwise.} \end{cases}$$

The coefficient associated to two components E and F is:

$$\begin{cases} E \cdot F & \text{if } E = F \\ \sum_e \chi(e) & \text{if } E \neq F. \end{cases}$$

where the sum runs along all the oriented edges from E to F .

Theorem 3.8 ([2]). *Let \mathcal{A} be an arrangement, and let ξ be a torsion character on $\pi_1(E_{\mathcal{A}})$ then:*

$$\overline{\operatorname{depth}}(\xi) = \operatorname{corank}(A_{\xi}).$$

The description of the inclusion map done in [11] (see also Section 4) and the result on the invariant obtain in Section 2 allow to compute explicitly the quasi-projective depth of any torsion character.

Example 3.9. In the case of the arrangement \mathcal{C}_7 with the character ξ defined in Example 2.7, the matrix A_ξ is:

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \chi_{(\mathcal{C}_7, \xi)}(\gamma) \\ 1 & \chi_{(\mathcal{C}_7, \xi)}(\gamma)^{-1} & -1 \end{pmatrix}.$$

Example 2.7 implies that $\overline{\text{depth}}(\xi) = 2$. Remark that this result is in harmony with the one obtained by D. Cohen and A. Suciu in [8].

4. COMPUTATION OF THE INVARIANT

The definition of $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is quite clear, and most probably it may have a more algebraic description. Nevertheless, its actual definition is topological, and we need concrete models of the topology of \mathcal{A} and more specifically of the embedding $B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$ at a homology level, see [11, 12].

In this section we compute the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ from a *wiring diagram* of \mathcal{A} . As we can see below, the wiring diagram is a $(3, 1)$ -dimensional model of the pair $(\mathbb{P}^2, \mathcal{A})$ which contains all the multiple points of \mathcal{A} . Given a cycle γ in $\Gamma_{\mathcal{A}}$, we will use the diagram to construct a specific nearby cycle $\tilde{\gamma}$ and calculate its value by the character ξ . This gives a general method to compute the invariant from the equations of \mathcal{A} .

Let us fix an arbitrary line $\ell_0 \in \mathcal{A}$ which will be considered as the line at infinity.

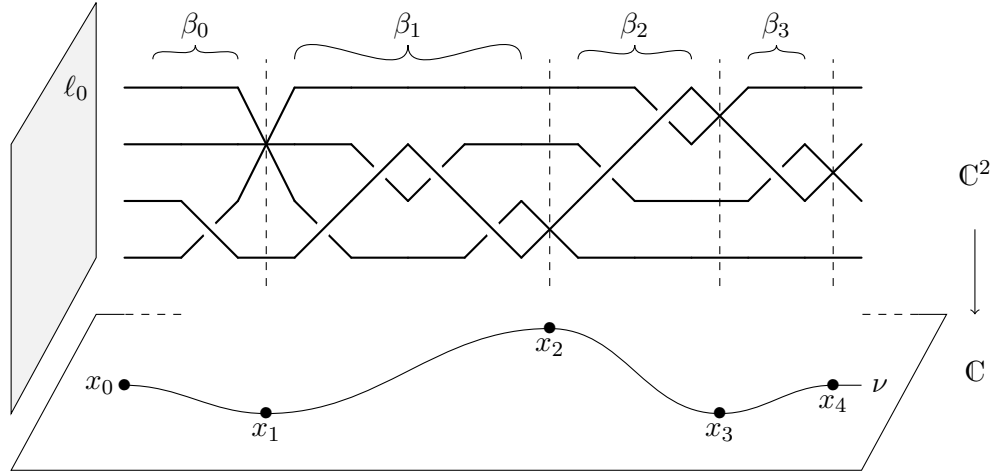


FIGURE 2. Decomposition of a wiring diagram.

Consider the affine arrangement $\mathcal{A}_0 := \mathcal{A} \setminus \ell_0$ of $\mathbb{C}^2 \equiv \mathbb{P}^2 \setminus \ell_0$; for a projective line $\ell \in \mathcal{A}$, $\ell \neq \ell_0$ we will denote by L the corresponding affine line $\ell \setminus \ell_0$. Let $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a linear projection, *generic* with respect to \mathcal{A} in the sense that for all $i \in \{1, \dots, n\}$, the restriction of $\pi|_{L_i}$ is a homeomorphism. We choose the coordinates in \mathbb{C}^2 such that π is the first projection. Suppose that the multiple points of \mathcal{A} lie in different fibers of π and that their images x_i satisfy $\Re(x_1) < \dots < \Re(x_k)$.

Consider a smooth path $\nu : [0, 1] \rightarrow \mathbb{C}$ whose image starts from a regular value $x_0 \in \pi(\text{Tub}(L_0))$ and passes through x_1, \dots, x_k in order with $\nu(t_j) = x_j$. The *braided wiring diagram* $W_{\mathcal{A}}$ associated to ν (voir [5]) is defined by:

$$W_{\mathcal{A}} = \left\{ (t, y) \in [0, 1] \times \mathbb{C} \mid (\nu(t), y) \in \bigcup \mathcal{A}_0 \right\}.$$

The space $W_{\mathcal{A}} \subset [0, 1] \times \mathbb{C}$ is a singular braid with n strings labelled according to the lines, whose singular points correspond to the multiple points of \mathcal{A} . Let us fix a generic projection $\pi_{\mathbb{R}} : \mathbb{C} \rightarrow \mathbb{R}$ such that the n strings are generic outside the singular points. For $u = 0, \dots, k-1$, the wiring diagram over (t_u, t_{u+1}) is identified with a regular braid β_u in the braid group B_n with n strands, see Figure 2.

Associated to any singular fiber $\pi^{-1}(t_u)$, containing the multiple point $P = L_{p_1} \cap \dots \cap L_{p_{s(P)}}$, let τ_u be the local positive half-twist between the strings $p_1, \dots, p_{s(P)}$ leaving straight the other strings (note that those strings are consecutive).

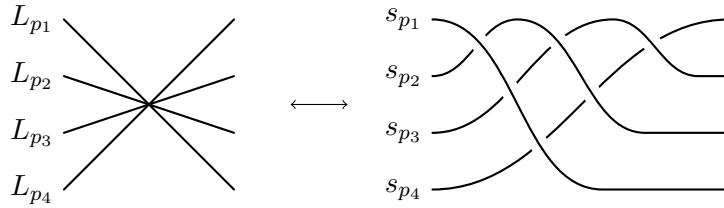


FIGURE 3. The half-twist associated to a singular point of multiplicity 4.

For all $u, v \in \{0, \dots, k\}$ with $u \neq v$, define

$$\beta_{u,v} = \begin{cases} \beta_{v-1} \cdot \tau_{v-1} \cdot \beta_{v-2} \cdot \dots \cdot \beta_{u+1} \cdot \tau_{u+1} \cdot \beta_u & u < v \\ (\beta_{v,u})^{-1} & v < u \end{cases}.$$

The braid $\beta_{u,v}$ is obtained from $W_{\mathcal{A}}$ by taking the sub-singular braid bounded by the singular points P_u and P_v and replacing the singular crossings by the corresponding local half-twist τ . It should be noted that the braided wiring diagram encodes in fact the braid monodromy of \mathcal{A} relative to the projection π . The operation of replacing singular crossings by half twists corresponds to a particular choice of a geometric basis of $\pi_1(\mathbb{C} \setminus \{x_1, \dots, x_k\})$, obtained from perturbations of the path ν .

For any braid $\beta \in B_n$ whose strings are labelled by s_1, \dots, s_n and all $k, l = 1, \dots, n$ with $k \neq l$, let $a_{k,l}(\beta)$ be defined as follows. If p is a crossing of β between s_k and s_l , let $c_{k,l}(p) = 1$ if the string s_k

goes over the string s_l and 0 otherwise. Let $w(p)$ be the sign of the crossing. Hence

$$a_{k,l}(\beta) = \left(\sum_p w(p) c_{k,l}(p) \right) \cdot v_k \in H_1(E_{\mathcal{A}}),$$

where the sum is taken over all the crossings of β between the strings s_k and s_l .

Remark 4.1. The number $a_{k,l}(\beta)$ is the algebraic sum of crossings in β between s_k and s_l where s_k goes over s_l .

The cycle γ of $\Gamma_{\mathcal{A}}$ can be described by a cyclically ordered sequence of line-vertices $v_{\ell_{i_0}}, \dots, v_{\ell_{i_r}}$. Up to re-ordering the lines, suppose that $i_0 = 0$ (this will simplify the computation of the invariant, though it is not necessary). Let j_q be the sequence of the indices of the point-vertices of γ , such that $\ell_{i_q} \cap \ell_{i_{q+1}} = P_{j_q}$.

Through the wiring diagram $W_{\mathcal{A}}$, the cycle γ can be seen directly in \mathcal{A} , in a unique way, by identifying its vertices to their corresponding wires between two multiple points. Since $\ell_{i_0} = \ell_0$ is the line at infinity, we obtain a cycle in $W_{\mathcal{A}}$ relative to the fiber over x_0 , uniquely defined. Indeed, one starts from a point in $\pi^{-1}(x_0)$, along the string labelled by s_{i_1} , we go to the singular point containing both L_{i_1} and L_{i_2} ; then along the string s_{i_2} , we go to the singular point containing both L_{i_2} and L_{i_3} and so on. We finish by going back to $\pi^{-1}(x_0)$ along the string s_{i_r} .

This process *almost* produce an embedding $r(\gamma)$ as in Definition 1.10. Such an embedding can be obtained by avoiding the singular points in the wires corresponding to points p not in the cycle. In order to construct the nearby cycle $\tilde{\gamma}$ we follow the procedure in [11, Section 4] where the *pushed* cycle in the boundary manifold $B_{\mathcal{A}}$ is called a *framed cycle*. If the wire of some L_{i_j} pass through a point $p \neq P_{j_q}$, then the part of framed cycle in $\partial \mathbb{B}_p$ is in the boundary of a regular neighborhood of the trivial knot $L_{i_j} \cap \partial \mathbb{B}_p$ in $\partial \mathbb{B}_p$. Hence the framed cycle $\tilde{\gamma}$ is a nearby cycle.

It is worth noticing that the way to push γ off \mathcal{A} depends on conventions, in particular around the multiple points of \mathcal{A} different from P_{j_q} . Different conventions might give another nearby cycle, but whose the image by the character ξ depends only on γ .

The next step is to compute the image of $\tilde{\gamma}$ by i_* induced by the inclusion map $B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$ at homology level. We use an abelian version of [11, Theorem 4.3], given in Lemma 4.2.

Lemma 4.2. *Let $s, t \in \{1, \dots, n\}$ with $s \neq t$, and $\mathfrak{e}_{s,t}$ be the cycle of $\Gamma_{\mathcal{A}}$ defined by v_{L_0}, v_{L_s} and v_{L_t} . Suppose that the intersection $L_s \cap L_t$ lies in the fiber $\pi^{-1}(x_u)$. There exists a nearby cycle $\tilde{\mathfrak{e}}_{s,t}$ associated with $\mathfrak{e}_{s,t}$ such that the image by $i_* : H_1(B_{\mathcal{A}}) \rightarrow H_1(E_{\mathcal{A}})$ induced by the inclusion is*

$$i_*(\tilde{\mathfrak{e}}_{s,t}) = \sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\beta_{0,u}) + \sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\beta_{u,0}) \in H_1(E_{\mathcal{A}}).$$

We postpone the proof of Lemma 4.2 to the end of the section. As a direct consequence, the image of $\tilde{\gamma}$ by the character ξ is given in the next lemma.

Proposition 4.3. *Let $\tilde{\gamma}$ be a nearby cycle for an inner cyclic realization $(\mathcal{A}, \xi, \gamma)$. With the previous notations, we have:*

$$\chi_{(\mathcal{A}, \xi)}(\tilde{\gamma}) = \xi \left(\sum_{q=1}^r \sum_{\substack{j=1, \\ j \neq i_q}}^n a_{j, i_q}(\beta_{t_{q-1}, t_q}) \right).$$

Proof. For all $i, j = 1, \dots, n$ with $i \neq j$, consider the cycles $\mathbf{e}_{i,j}$ given by the sequence $v_{L_0}, v_{L_i}, v_{L_j}$, as in Lemma 4.2. They generate $H_1(\Gamma_{\mathcal{A}})$ and γ can be decomposed as $\gamma = \mathbf{e}_{i_1, i_2} + \mathbf{e}_{i_2, i_3} + \dots + \mathbf{e}_{i_{r-1}, i_r}$. For any choice of nearby cycles $\tilde{\mathbf{e}}_{i,j}$ associated with $\mathbf{e}_{i,j}$, the sum $\tilde{\gamma} = \tilde{\mathbf{e}}_{i_1, i_2} + \dots + \tilde{\mathbf{e}}_{i_{r-1}, i_r}$ is a nearby cycle associated with γ . Remark that Lemma 2.2 implies that the image of $i_*(\tilde{\gamma})$ by ξ does not depend on the choice of $\tilde{\gamma}$. From Lemma 4.2, we have:

$$\xi \circ i_*(\tilde{\mathbf{e}}_{i_q, i_{q+1}}) = \xi \left(\sum_{\substack{j=1, \\ j \neq i_q}}^n a_{j, i_q}(\beta_{0, t_q}) + \sum_{\substack{j=1, \\ j \neq i_{q+1}}}^n a_{j, i_{q+1}}(\beta_{t_q, 0}) \right).$$

Since β_{0, t_q} and $\beta_{0, t_{q+1}}$ are equal along $\beta_{0, \min(t_q, t_{q+1})}$, then we have:

$$\sum_{\substack{j=1, \\ j \neq i_{q+1}}}^n a_{j, i_{q+1}}(\beta_{t_q, 0}) + \sum_{\substack{j=1, \\ j \neq i_{q+1}}}^n a_{j, i_{q+1}}(\beta_{0, t_{q+1}}) = \sum_{\substack{j=1, \\ j \neq i_{q+1}}}^n a_{j, i_{q+1}}(\beta_{t_q, t_{q+1}}) + \sum_{q=0}^m \theta_{i_q, i_{q+1}} + \theta'_{t_q},$$

where θ'_{t_q} is a product of meridians associated with lines in the support (and then in $\ker(\xi)$) coming from the change of wires at the point P_{t_q} . Finally, we have:

$$i_*(\tilde{\gamma}) = \sum_{q=1}^r \sum_{\substack{j=1, \\ j \neq i_q}}^n a_{j, i_q}(\beta_{t_q, t_{q+1}}) + \Theta,$$

where Θ is an element of $\ker(\xi)$. □

Proof of Lemma 4.2. As mentioned above, we *push* a model of $\mathbf{e}_{s,t}$ from \mathcal{A} to the boundary manifold $B_{\mathcal{A}}$. The cycle obtained is a *framed* cycle. To compute the image of this particular nearby cycle, we express it as a product of meridians and a *geometric* cycle $\mathcal{E}_{s,t}$.

By construction, geometric cycles bound 2-cells with holes in the exterior $E_{\mathcal{A}}$ and counting intersections allows a direct computation of $i_* : H_1(B_{\mathcal{A}}) \rightarrow H_1(E_{\mathcal{A}})$ in this basis. The previous expression can be obtain, by [11, Proposition 4.1], using the unknotting map δ (sending framed cycle on the corresponding geometric cycle). The contribution of each singular point P correspond to the contribution

of the strings overcrossing the string s and the string t in τ_P , then an abelian computation shows that:

$$\delta(\tilde{\mathbf{e}}_{s,t}) = \mathcal{E}_{s,t} = - \sum_{q=1}^{m-1} \left(\sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\tau_{P_q}) \right) + \tilde{\mathbf{e}}_{s,t} + \sum_{q=1}^{m-1} \left(\sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\tau_{P_q}) \right).$$

The model of $\mathbf{e}_{s,t}$ in $W_{\mathcal{A}}$ is composed of two parts: a string labelled s from $\pi^{-1}(x_0)$ to $L_s \cap L_t \in \pi^{-1}(x_u)$, and a string labelled t from $\pi^{-1}(x_u)$ to $\pi^{-1}(x_0)$. Let S_s (resp. S_t) be the set of arcs that go over the string s (resp. t) between $\pi^{-1}(x_0)$ and $\pi^{-1}(x_u)$. For each arc ς in S_s or S_t , let $\text{sgn}(\varsigma) \in \{\pm 1\}$ be the sign of the base $\{\varsigma, \mathbf{e}_{s,t}\}$ (ς is oriented from left to right in $W_{\mathcal{A}}$) where ς and $\mathbf{e}_{s,t}$ are considered as vectors in \mathbb{R}^2 . Let $v_{\varsigma} \in H_1(E_{\mathcal{A}})$ be the meridian of the line corresponding to ς . Once again, an abelian computation shows that:

$$i_*(\mathcal{E}_{s,t}) = \mu_{s,t} = \sum_{\varsigma \in S_s} \text{sgn}(\varsigma) v_{\varsigma} + \sum_{\varsigma \in S_t} \text{sgn}(\varsigma) v_{\varsigma} = \sum_{q=1}^u \sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\beta_{q-1,q}) + \sum_{q=1}^u \sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\beta_{q-1,q}^{-1}).$$

Finally, we obtain

$$\begin{aligned} i_*(\tilde{\mathbf{e}}_{s,t}) &= \sum_{q=1}^{u-1} \left(\sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\tau_{P_q}) \right) + \sum_{q=1}^u \left(\sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\beta_{q-1,q}) \right) + \sum_{q=1}^u \left(\sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\beta_{q-1,q}^{-1}) \right) + \sum_{q=1}^{u-1} \left(\sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\tau_{P_q}^{-1}) \right), \\ &= \sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\beta_{0,1} \cdot \tau_{P_1} \cdots \tau_{P_{u-1}} \cdot \beta_{u-1,u}) + \sum_{\substack{j=1, \\ j \neq s}}^n a_{j,t}(\beta_{u-1,u}^{-1} \cdot \tau_{P_{u-1}}^{-1} \cdots \tau_{P_1}^{-1} \cdot \beta_{i_0,i_1}^{-1}), \\ &= \sum_{\substack{j=1, \\ j \neq s}}^n a_{j,s}(\beta_{0,u}) + \sum_{\substack{j=1, \\ j \neq t}}^n a_{j,t}(\beta_{u,0}). \end{aligned} \quad \square$$

5. EXAMPLE

The MacLane arrangements are two conjugated arrangements coming from MacLane's matroid [14]. It is the arrangement with a minimal number of lines such that the combinatorics admits a realization over \mathbb{C} but not over \mathbb{R} (see [6, Example 6.6.2(3)]). These arrangements are constructed as follows. Let us consider the 2-dimensional vector space on the field \mathbb{F}_3 of three elements. Such a plane contains 9 points and 12 lines, 4 of them pass through the origin. Let \mathcal{L} be $\mathbb{F}_3^2 \setminus \{(0,0)\}$ and \mathcal{P} the set of lines in \mathbb{F}_3^2 . This provides a line combinatorics $(\mathcal{L}, \mathcal{P}, \subseteq)$, where for all $\ell \in \mathcal{L}, P \in \mathcal{P}$, we have $P \subseteq \ell \Leftrightarrow (\ell \in P, \text{ in } \mathbb{F}_3^2)$. Figure 4 represents the ordered MacLane's combinatorics viewed in \mathbb{F}_3^2 . As an ordered combinatorics, it admits two ordered complex realizations.

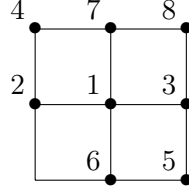
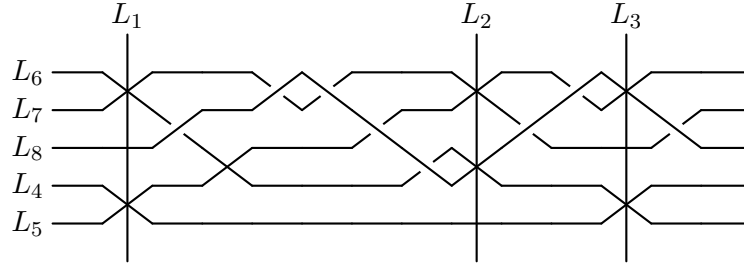
FIGURE 4. Ordered MacLane combinatorics, viewed in \mathbb{F}_3^2 

FIGURE 5. Wiring diagram of extended MacLane arrangement

We can give equations to the realizations:

$$\begin{aligned} L_1 : y - \zeta z = 0, \quad L_2 : y - z = 0, \quad L_3 : y - \bar{\zeta} z = 0, \quad L_4 : x - z = 0, \\ L_5 : x - \bar{\zeta} y = 0, \quad L_6 : x - \zeta y = 0, \quad L_7 : x - \bar{\zeta} z = 0, \quad L_8 : x - \zeta z = 0, \end{aligned}$$

where ζ is a primitive cubic root of unity (its choice determine the realization). Add to MacLane arrangements a line L_0 passing through the intersection points: $L_1 \cap L_2 \cap L_3$ and $L_4 \cap L_7 \cap L_8$, i.e., $L_0 : z = 0$ in the above equations. We obtain two ordered realization denoted by \mathcal{M}^+ and \mathcal{M}^- and called respectively positive and negative extended MacLane arrangement (the wiring diagram of \mathcal{M}^+ is pictured in Figure 5). It is not hard to see that the only inner-cyclic characters are ξ and ξ^{-1} where ξ is defined by:

$$(v_0, v_1, \dots, v_8) \mapsto (1, \zeta, \zeta, \zeta, \zeta^2, 1, 1, \zeta^2, \zeta^2),$$

with corresponding cycle $\gamma : v_{L_0} \rightarrow v_{P_{0,6}} \rightarrow v_{L_6} \rightarrow v_{P_{5,6}} \rightarrow v_{L_5} \rightarrow v_{P_{0,5}} \rightarrow v_{L_0}$ in $\Gamma_{\mathcal{M}^+}$. With the notation of Section 4, the cycle $\tilde{\mathbf{e}}_{6,5}^\pm$ is a nearby cycle associated with γ . In the positive case, the braid $\beta^+ \in B_8$ associated with $\tilde{\mathbf{e}}_{6,5}^+$ is $(\sigma_5 \sigma_4 \sigma_5) \sigma_3 (\sigma_2 \sigma_1 \sigma_2) \sigma_4^{-1}$, while in the negative $\beta^- = (\sigma_5 \sigma_4 \sigma_5) \sigma_3 (\sigma_2 \sigma_1 \sigma_2) \sigma_4$. Then

using Proposition 4.3, the images in the complement of the nearby cycle in both cases are:

$$\begin{aligned} i_*(\tilde{\mathfrak{e}}_{6,5}^+) &= \sum_{\substack{j=1, \\ j \neq 6}}^8 a_{j,6}(\beta^+) + \sum_{\substack{j=1, \\ j \neq 5}}^8 a_{j,5}((\beta^+)^{-1}) = (v_1 - v_8) + (-v_4 - v_1) = -v_8 - v_4, \\ i_*(\tilde{\mathfrak{e}}_{6,5}^-) &= \sum_{\substack{j=1, \\ j \neq 6}}^8 a_{j,6}(\beta^-) + \sum_{\substack{j=1, \\ j \neq 5}}^8 a_{j,5}((\beta^-)^{-1}) = v_1 + (-v_4 - v_1) = -v_4. \end{aligned}$$

Finally, we obtain that :

$$\mathcal{I}(\mathcal{M}^+, \xi, \gamma) = \zeta^2, \quad \mathcal{I}(\mathcal{M}^-, \xi, \gamma)(\gamma) = \zeta.$$

As a consequence of this paper there is no homeomorphism $\phi : (\mathbb{P}^2, \mathcal{M}^+) \rightarrow (\mathbb{P}^2, \mathcal{M}^-)$ preserving orders and orientations of the lines. Note that it is a consequence that this result is already true for the MacLane arrangements; the fact that MacLane arrangements satisfy this property is done using the techniques in [4, 16, 17]) and the invariant in this paper does not provide an obstruction.

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